Pseudo-Hermiticity and Electromagnetic Wave Propagation: The case of anisotropic and lossy media

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Abstract

Pseudo-Hermitian operators can be used in modeling electromagnetic wave propagation in stationary lossless media. We extend this method to a class of non-dispersive anisotropic media that may display loss or gain. We explore three concrete models to demonstrate the utility of our general results and reveal the physical meaning of pseudo-Hermiticity and quasi-Hermiticity of the relevant wave operator. In particular, we consider a uniaxial model where this operator is not diagonalizable. This implies left-handedness of the medium in the sense that only clockwise circularly polarized plane-wave solutions are bounded functions of time.

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1 Introduction

Consider a stationary (time-independent) dispersionless and source-free medium with permittivity and permeability tensors $\overleftrightarrow{\varepsilon} = \overleftrightarrow{\varepsilon}(\vec{x})$ and $\overleftrightarrow{\mu} = \overleftrightarrow{\mu}(\vec{x})$. Then the Maxwell's equations are given by [1]:

$$\vec{\nabla} \cdot (\overleftrightarrow{\varepsilon}\vec{E}) = \vec{\nabla} \cdot \vec{B} = 0, \tag{1}$$

$$\dot{\vec{E}} = \overleftrightarrow{\varepsilon}^{-1}\mathfrak{D}\overleftrightarrow{\mu}^{-1}\vec{B}, \tag{2}$$

$$\dot{\vec{B}} = -\mathfrak{D}\vec{E}, \tag{3}$$

where an overdot stands for a time-derivative, and $\mathfrak{D}$ is the curl operator, e.g., $\mathfrak{D}\vec{E} := \vec{\nabla} \times \vec{E}$. As shown in Ref. [2], Eqs. (2) and (3) are equivalent to

$$\vec{B}(\vec{r}, t) = \vec{B}_0(\vec{r}) - \int_0^t \mathfrak{D}\vec{E}(\vec{r}, \tau)\, d\tau, \tag{4}$$

$$\ddot{\vec{E}} + \Omega^2\vec{E} = 0, \tag{5}$$

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where $\vec{B}_0(\vec{r}) := \vec{B}(\vec{r},0)$ and

$$\Omega^2 := \overleftrightarrow{\varepsilon}^{-1}\mathfrak{D}\,\overleftrightarrow{\mu}^{-1}\mathfrak{D}. \tag{6}$$

We can view $\mathfrak{D}$ as a linear operator acting in the Hilbert space $\mathcal{H}$ of square-integrable complex vector fields $\vec{F}:\mathbb{R}^3 \to \mathbb{C}$. This is defined by the L^2-inner product,

$$\prec \vec{F}|\vec{G} \succ := \int_{\mathbb{R}^3} \vec{F}(\vec{x})^* \cdot \vec{G}(\vec{x})\; d\vec{x}^3,$$

where a dot placed between $\vec{F}^*$ and $\vec{G}$ stands for the usual dot product.[1]

It is easy to see that $\mathfrak{D}$ is a Hermitian operator acting in $\mathcal{H}$. Furthermore, for the cases that $\overleftrightarrow{\varepsilon}$ and $\overleftrightarrow{\mu}$ are given by invertible Hermitian matrices, they also define invertible Hermitian operators acting in $\mathcal{H}$. This in turn implies that Ω^2 satisfies the pseudo-Hermiticity relation [3]:

$$\Omega^{2\dagger} = \overleftrightarrow{\varepsilon}\,\Omega^2\overleftrightarrow{\varepsilon}^{-1}. \tag{7}$$

In particular, for a lossless (and gainless) medium $\overleftrightarrow{\varepsilon}$ is a positive-definite operator, and Ω^2 is quasi-Hermitian [4], i.e., it is related to a Hermitian operator via a similarity transformation [5]. This observation suggests an interesting spectral method of solving the wave equation (5). Ref. [2] gives a rather explicit application of this method for an effectively one-dimensional model involving localized inhomogeneity. Ref. [6] studies the possibility of extending this method to a class of lossless dispersive media.

The purpose of the present article is to use the spectral properties of the operator Ω^2 in modeling electromagnetic wave propagation in stationary non-dispersive anisotropic media that display loss or gain. This requires considering non-Hermitian permittivity and permeability tensors [7, 8]. In this case, Ω^2 is generally no longer quasi-Hermitian, and a direct application of the spectral method outlined in Ref. [2] is intractable. In the following we will examine the structure of Ω^2 for a lossy anisotropic medium. Our investigation is motivated by the well-known fact that in many physical applications one deals with such materials. A typical example is a magnetic photonic crystals [9].[2]

2 Solution of the Wave Equation

The wave equation (5) admits the following formal solution [2]

$$\vec{E}(\vec{r},t) = \cos(\Omega t)\vec{E}_0(\vec{r}) + \Omega^{-1}\sin(\Omega t)\dot{\vec{E}}_0(\vec{r}), \tag{8}$$

where $\vec{E}_0(\vec{r}) := \vec{E}(\vec{r},0)$, $\dot{\vec{E}}_0(\vec{x}) := \dot{\vec{E}}(\vec{r},0)$, and

$$\cos(\Omega t) := \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\,(t^2\Omega^2)^n, \qquad \Omega^{-1}\sin(\Omega t) := t\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,(t^2\Omega^2)^n. \tag{9}$$

In this section we will study the structure of the solution (8) for the cases that the permittivity and permeability tensors, $\overleftrightarrow{\varepsilon}$ and $\overleftrightarrow{\mu}$, are given by constant non-Hermitian 3×3 matrices. We achieve this by performing a Fourier transform.

[1] For all $\vec{F}\in\mathcal{H}$, $\prec \vec{F}|\vec{F}\succ < \infty$. This in particular implies that $|\vec{F}(\vec{x})| \to 0$ as $|\vec{x}|\to\infty$.

[2] See [10] for a comprehensive discussion of lossy crystals.

In the Fourier basis the curl operator $\mathfrak{D}$ is represented by the antisymmetric Hermitian matrix:

$$\mathfrak{D} = \left(\begin{array}{ccc} 0 & ik_3 & -ik_2 \\ -ik_3 & 0 & ik_1 \\ ik_2 & -ik_1 & 0 \end{array} \right), \qquad k_1, k_2, k_3 \in \mathbb{R}, \tag{10}$$

while $\overleftrightarrow{\varepsilon}^{-1}$ and $\overleftrightarrow{\mu}^{-1}$ that appear in the expression (6) for Ω^2 are 3×3 complex matrices.[3] It is not difficult to compute Ω^2 in the Fourier basis. This gives rise to a 3×3 matrix involving k_i and the entries of $\overleftrightarrow{\varepsilon}$ and $\overleftrightarrow{\mu}$, i.e., a total of 39 real parameters. Rather than giving this complicated expression, we will examine the Jordan canonical form of Ω^2. In fact, it is more convenient to work with the dimensionless operator,

$$\hat{\Omega}^2 := \omega_0^{-2}\Omega^2 = \omega_0^{-2}\,\overleftrightarrow{\varepsilon}^{-1}\mathfrak{D}\,\overleftrightarrow{\mu}^{-1}\mathfrak{D}, \tag{11}$$

where

$$\omega_0 := c\,|\vec{k}| = c\sqrt{k_1^2 + k_2^2 + k_3^2}\,, \qquad \vec{k} := \left(\begin{array}{c} k_1 \\ k_2 \\ k_3 \end{array} \right). \tag{12}$$

In terms of $\hat{\Omega}^2$, (8) and (9) read

$$\vec{E}(\vec{r}, t) = \cos(\hat{\Omega}\,\omega_0 t)\vec{E}_0(\vec{r}) + \omega_0^{-1}\hat{\Omega}^{-1}\sin(\hat{\Omega}\,\omega_0 t)\dot{\vec{E}}_0(\vec{r}), \tag{13}$$

$$\cos(\hat{\Omega}\,\omega_0 t) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\,(\omega_0^2 t^2 \hat{\Omega}^2)^n, \tag{14}$$

$$\hat{\Omega}^{-1}\sin(\hat{\Omega}\,\omega_0 t) := \omega_0 t \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\,(\omega_0^2 t^2 \hat{\Omega}^2)^n. \tag{15}$$

A straightforward consequence of (10) is $\mathfrak{D}\vec{k} = \vec{0}$. Therefore, $\mathfrak{D}$ has a zero eigenvalue and owing to its symmetry a pair of real eigenvalues with opposite sign, namely $\pm|\vec{k}|$. In view of (11), this implies that $\hat{\Omega}^2$ will also have a zero eigenvalue. Hence, its Jordan canonical form J_{Ω^2} must have one of the following two forms.

$$\text{Case 1}: \qquad J_{\Omega^2} = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & \lambda_- & 0 \\ 0 & 0 & \lambda_+ \end{array} \right), \qquad \lambda_\pm \in \mathbb{C}, \tag{16}$$

$$\text{Case 2}: \qquad J_{\Omega^2} = \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{array} \right), \qquad \lambda \in \mathbb{C}. \tag{17}$$

These correspond to diagonalizable and non-diagonalizable cases, respectively.

[3] Given a linear operator acting in $\mathcal{H}$, its representation in the Fourier basis, $\tilde{L}$, is related to its representation in the position basis, L, according to: $\tilde{L}\tilde{\vec{F}}(\vec{k}) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-i\vec{k}\cdot\vec{x}} L\vec{F}(\vec{x})\,d\vec{x}^3$. Here $\vec{F}$ is an arbitrary complex vector field having a Fourier transform $\tilde{\vec{F}}$ defined by $\tilde{\vec{F}}(\vec{k}) := (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{-i\vec{k}\cdot\vec{x}}\vec{F}(\vec{x})\,d\vec{x}^3$. Throughout this article we use the same symbol for the representations of an operator in both Fourier and position bases to simplify the notation. This convention does not lead to any confusion, for we exclusively use the Fourier representation of the relevant operators.

Let S be an invertible matrix fulfilling $\hat{\Omega}^2 = S^{-1} J_{\Omega^2} S$. Then, in view of (14) and (15),

$$\cos(\omega_0 t\hat{\Omega}) = S^{-1}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\left(\omega_0^2 t^2 J_{\Omega^2}\right)^n\right)S, \tag{18}$$

$$\hat{\Omega}^{-1}\sin(\omega_0 t\hat{\Omega}) = S^{-1}\left(\omega_0 t\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\left(\omega_0^2 t^2 J_{\Omega^2}\right)^n\right)S. \tag{19}$$

In order to simplify these equations we consider Cases 1 and 2 separately.

Case 1 ($\hat{\Omega}^2$ is diagonalizable): In this case,

$$(J_{\Omega^2})^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda_-^n & 0 \\ 0 & 0 & \lambda_+^n \end{pmatrix} \qquad \text{for all } n \in \mathbb{Z}^+.$$

Inserting this relation in (18) and (19) yields

$$\cos(\omega_0 t\hat{\Omega}) = S^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\sqrt{\lambda_-}\,\omega_0 t) & 0 \\ 0 & 0 & \cos(\sqrt{\lambda_+}\,\omega_0 t) \end{pmatrix} S, \tag{20}$$

$$\hat{\Omega}^{-1}\sin(\omega_0 t\hat{\Omega}) = S^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda_-}}\sin(\sqrt{\lambda_-}\,\omega_0 t) & 0 \\ 0 & 0 & \frac{1}{\sqrt{\lambda_+}}\sin(\sqrt{\lambda_+}\,\omega_0 t) \end{pmatrix} S \tag{21}$$

Note that the time-harmonic solutions of the wave equation (13) are the eigenvectors of $\hat{\Omega}^2$, [2, 6]. Hence, they are given by

$$\vec{E}^{(a)}_{\pm}(\vec{r},t) = \int_{\mathbb{R}^3} d^3k\,\left[\mathscr{F}_{\pm}(\vec{k})e^{i(\vec{k}\cdot\vec{r}-\sqrt{\lambda_{\pm}}\,\omega_0 t)} + \mathscr{G}_{\pm}(\vec{k})e^{i(\vec{k}\cdot\vec{r}+\sqrt{\lambda_{\pm}}\,\omega_0 t)}\right] S^{-1}\hat{e}_a, \qquad a = 1,2, \tag{22}$$

where $\mathscr{F}_{\pm}$ and $\mathscr{G}_{\pm}$ are complex-valued functions, and

$$\hat{e}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \hat{e}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Clearly, for $a = 1$ and $a = 2$, $S^{-1}\hat{e}_a$ is the second and the third column of the matrix S^{-1}, respectively. Furthermore, the matrix S and the eigenvalues $\lambda_{\pm}$ that enter (22) are, in general, functions of $\vec{k}$.

Case 2 ($\hat{\Omega}^2$ is non-diagonalizable): In this case,

$$(J_{\Omega^2})^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{pmatrix}, \qquad \text{for all } n \in \mathbb{Z}^+, \tag{23}$$

and we find

$$\cos(\omega_0 t\hat{\Omega}) = S^{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\sqrt{\lambda}\,\omega_0 t) & -\frac{\omega_0 t}{2\sqrt{\lambda}}\sin(\sqrt{\lambda}\,\omega_0 t) \\ 0 & 0 & \cos(\sqrt{\lambda}\,\omega_0 t) \end{pmatrix} S, \tag{24}$$

$$\hat{\Omega}^{-1}\sin(\omega_0 t\hat{\Omega}) = S^{-1}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\,\omega_0 t) & \frac{1}{2\lambda}\left[\omega_0 t\cos(\sqrt{\lambda}\,\omega_0 t) - \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\,\omega_0 t)\right] \\ 0 & 0 & \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}\,\omega_0 t) \end{pmatrix} S. \tag{25}$$

Moreover, the time-harmonic solutions of the wave equation (13) take the form

$$\vec{E}_{\pm}(\vec{r},t) = \int_{\mathbb{R}^3} d^3k\,\left[\mathscr{F}(\vec{k})e^{i(\vec{k}\cdot\vec{r}-\sqrt{\lambda}\,\omega_0 t)} + \mathscr{G}(\vec{k})e^{i(\vec{k}\cdot\vec{r}+\sqrt{\lambda}\,\omega_0 t)}\right] S^{-1}\hat{e}_2. \tag{26}$$

where $\mathscr{F}$ and $\mathscr{G}$ are complex-valued functions.

We close this section by revealing an implicit symmetry of the operator Ω^2. Here we will momentarily consider the possibility that permittivity and permeability tensors are not constant.

As we have seen, Ω^2 admits a single zero eigenvalue. The presence of this zero eigenvalue implies the existence of parts of $\overleftrightarrow{\varepsilon}^{-1}$ and $\overleftrightarrow{\mu}^{-1}$ that have no influence on the propagating electric field. In order to characterize these, we expand $\overleftrightarrow{\varepsilon}^{-1}$ and $\overleftrightarrow{\mu}^{-1}$ as follows:

$$\overleftrightarrow{\varepsilon}^{-1} = \overleftrightarrow{\varepsilon}'_{H_0} + \overleftrightarrow{\varepsilon}'_{H_1} + \overleftrightarrow{\varepsilon}'_{A_0} + \overleftrightarrow{\varepsilon}'_{A_1}, \tag{27}$$

$$\overleftrightarrow{\mu}^{-1} = \overleftrightarrow{\mu}'_{H_0} + \overleftrightarrow{\mu}'_{H_1} + \overleftrightarrow{\mu}'_{A_0} + \overleftrightarrow{\mu}'_{A_1}. \tag{28}$$

Here the subscripts H and A denote the "Hermitian" and "anti-Hermitian" parts of the tensors respectively,

$$\overleftrightarrow{\varepsilon}'_{H_1}\mathfrak{D} = \overleftrightarrow{\varepsilon}'_{A_1}\mathfrak{D} = \overleftrightarrow{\mu}'_{H_1}\mathfrak{D} = \mathfrak{D}\,\overleftrightarrow{\mu}'_{H_1} = \overleftrightarrow{\mu}'_{A_1}\mathfrak{D} = \mathfrak{D}\,\overleftrightarrow{\mu}'_{A_1} = \overleftrightarrow{0}, \tag{29}$$

and $\overleftrightarrow{0}$ stands for the 3×3 zero matrix.

Because the null space of Ω^2 is spanned by $\vec{k}$, according to (29) we have

$$\overleftrightarrow{\varepsilon}'_{H_1} = \alpha_\varepsilon(\vec{k})\,\vec{k}\vec{k}^\dagger,\quad \overleftrightarrow{\varepsilon}'_{A_1} = i\beta_\varepsilon(\vec{k})\,\vec{k}\vec{k}^\dagger,\quad \overleftrightarrow{\mu}'_{H_1} = \alpha_\mu(\vec{k})\,\vec{k}\vec{k}^\dagger,\quad \overleftrightarrow{\mu}'_{A_1} = i\beta_\mu(\vec{k})\,\vec{k}\vec{k}^\dagger, \tag{30}$$

where $\alpha_\varepsilon, \beta_\varepsilon, \alpha_\mu, \beta_\mu$ are real-valued functions of $\vec{k}$, and

$$\vec{k}\vec{k}^\dagger = \begin{pmatrix} k_1^2 & k_1k_2 & k_1k_3 \\ k_1k_2 & k_2^2 & k_2k_3 \\ k_1k_3 & k_2k_3 & k_3^2 \end{pmatrix}.$$

For the cases that

$$\overleftrightarrow{\varepsilon}^{-1}_{A_0} = \overleftrightarrow{\mu}^{-1}_{A_0} = \overleftrightarrow{0} \tag{31}$$

Ω^2 is pseudo-Hermitian. If in addition $\overleftrightarrow{\varepsilon}'_{H_0}$ is positive definite, then Ω^2 is quasi-Hermitian. Note that these hold irrespectively of the value of $\overleftrightarrow{\varepsilon}'_{H_1}$, $\overleftrightarrow{\varepsilon}'_{A_1}$, $\overleftrightarrow{\mu}'_{H_1}$, and $\overleftrightarrow{\mu}'_{A_1}$.

The above observation seems to be relevant only for the material with spatial dispersion [10]. For the non-dispersive material that we consider in this article, $\overleftrightarrow{\varepsilon}$ and $\overleftrightarrow{\mu}$ are independent of $\vec{k}$ and the above gauge freedom is effectively frozen (the scalar functions $\alpha_\varepsilon, \beta_\varepsilon, \alpha_\mu, \beta_\mu$ are identically zero).

3 Applications: Plane-Wave and Time-Harmonic Solutions

In this section we will examine the application of our general results in the study of three concrete models. This also allows for extracting the physical meaning of pseudo-Hermiticity and quasi-Hermiticity of the Ω^2.

Example 1: Consider the propagation of a plane wave given by the initial conditions

$$\vec{E}_0(x,y,z) = \mathcal{E}\left(\begin{array}{c}\cos\varphi\\ \sin\varphi\\ 0\end{array}\right)e^{ik_3 z}, \qquad \dot{\vec{E}}_0(x,y,z) = \vec{0}, \tag{32}$$

in a uniaxial medium with permittivity and permeability tensors [9]:

$$\overleftrightarrow{\varepsilon} = \varepsilon_0\left(\begin{array}{ccc}\varepsilon_1 + i\gamma_\varepsilon & i\alpha & 0\\ -i\alpha & \varepsilon_1 + i\gamma_\varepsilon & 0\\ 0 & 0 & \varepsilon_3\end{array}\right), \qquad \overleftrightarrow{\mu} = \mu_0\left(\begin{array}{ccc}\mu_1 + i\gamma_\mu & i\beta & 0\\ -i\beta & \mu_1 + i\gamma_\mu & 0\\ 0 & 0 & \mu_3\end{array}\right), \tag{33}$$

where $\mathcal{E} \in \mathbb{C}$ and $\varphi \in [0, 2\pi)$ determine the amplitude and polarization of the initial wave, ε_0 and μ_0 are the permittivity and permeability of the vacuum, so that $\varepsilon_0\mu_0 = 1/c^2$, and $\varepsilon_1, \varepsilon_2, \mu_1, \mu_2, \gamma_\varepsilon, \gamma_\mu, \alpha, \beta \in \mathbb{R}$ describe the electric and magnetic properties of the medium.[4]

A simple consequence of (32) is that in the calculation of $\hat{\Omega}^2$ in the Fourier basis we can set

$$k_1 = k_2 = 0. \tag{34}$$

This leads to an enormous simplification of the Jordan decomposition of $\hat{\Omega}^2$. In particular, $\hat{\Omega}^2$ is diagonalizable, $\omega_0 = c|k_3|$, and inserting (10) and (33) in (11) and calculating the eigenvalues of the resulting expression for $\hat{\Omega}^2$ we find

$$\lambda_\pm = [(\varepsilon_1 \pm \alpha + i\gamma_\varepsilon)(\mu_1 \pm \beta + i\gamma_\mu)]^{-1}. \tag{35}$$

Furthermore, the similarity transformation that diagonalizes $\hat{\Omega}^2$ is independent of the parameters of the system, for we have

$$S = \left(\begin{array}{ccc}0 & 0 & 2\\ i & 1 & 0\\ -i & 1 & 0\end{array}\right), \qquad S^{-1} = \frac{1}{2}\left(\begin{array}{ccc}0 & -i & i\\ 0 & 1 & 1\\ 1 & 0 & 0\end{array}\right). \tag{36}$$

Substituting (35) and (36) in (20) and using the resulting relation and (32) in (13), we obtain the following expression for the propagating electric field

$$\vec{E}(\vec{r}, t) = \vec{E}(z, t) = \mathcal{E}\vec{n}_E(t)\, e^{ik_3 z}, \tag{37}$$

where

$$\vec{n}_E(t) := \frac{1}{2}\left(\begin{array}{c}e^{-i\varphi}\cos(\sqrt{\lambda_-}\,\omega_0 t) + e^{i\varphi}\cos(\sqrt{\lambda_+}\,\omega_0 t)\\ i[e^{-i\varphi}\cos(\sqrt{\lambda_-}\,\omega_0 t) - e^{i\varphi}\cos(\sqrt{\lambda_+}\,\omega_0 t)]\\ 0\end{array}\right). \tag{38}$$

[4]In light of (2), $\dot{\vec{E}}_0(x,y,z) = \vec{0}$ follows from $\vec{B}_0(x,y,z) = \vec{0}$.

The expression for the magnetic field can be obtained using (4). Setting $\vec{B}(\vec{r},0)=\vec{0}$, which in view of (3) is consistent with the second equation in (32), and using (37) and (4), we find

$$\vec{B}(\vec{r},t)=i\,c^{-1}\mathcal{E}\,\vec{n}_B(t)\,e^{ik_3z}, \tag{39}$$

where

$$\vec{n}_B(t):=\frac{1}{2}\left(\begin{array}{c} i\left(\frac{e^{-i\varphi}\sin(\sqrt{\lambda_-}\,\omega_0t)}{\sqrt{\lambda_-}}-\frac{e^{i\varphi}\sin(\sqrt{\lambda_+}\,\omega_0t)}{\sqrt{\lambda_+}}\right)\\ -\left(\frac{e^{-i\varphi}\sin(\sqrt{\lambda_-}\,\omega_0t)}{\sqrt{\lambda_-}}+\frac{e^{i\varphi}\sin(\sqrt{\lambda_+}\,\omega_0t)}{\sqrt{\lambda_+}}\right)\\ 0\end{array}\right). \tag{40}$$

It is remarkable that for the cases where Ω^2 is a pseudo-Hermitian operator but not quasi-Hermitian, so that $\lambda_+=\lambda_-^*$, both the vectors $\vec{n}_E(t)$ and $\vec{n}_B(t)$ stay real. Therefore, for the model we consider, the propagating wave does not undergo a phase shift[5]; *the pseudo-Hermiticity of* Ω^2 *implies phase conservation.* In light of (35), $\lambda_+=\lambda_-^*$ and Ω^2 is pseudo-Hermitian (but not quasi-Hermitian) if and only if $\mu_1\alpha\neq 0$ and

$$\varepsilon_1\beta+\mu_1\alpha=0,\qquad \varepsilon_1\gamma_\mu+\mu_1\gamma_\varepsilon=0. \tag{41}$$

In view of (35), these relations imply

$$\lambda_\pm=\frac{\varepsilon_1}{\mu_1}\left[(\varepsilon_1^2+\gamma_\varepsilon^2-\alpha^2\mp i\alpha\gamma_\epsilon\right]^{-1}. \tag{42}$$

Next, we derive the necessary and sufficient conditions for the quasi-Hermiticity of Ω^2 (equivalently $\hat{\Omega}^2$). Because $\hat{\Omega}^2$ is diagonalizable, the latter is equivalent to the reality of the eigenvalues $\lambda_\pm$ of $\hat{\Omega}^2$. According to (35), $\lambda_\pm$ are real and Ω^2 is quasi-Hermitian if and only if either

$$\gamma_\epsilon=\gamma_\mu=0, \tag{43}$$

or the following two conditions hold:

$$\varepsilon_1\beta-\mu_1\alpha=0,\qquad \varepsilon_1\gamma_\mu+\mu_1\gamma_\varepsilon=0. \tag{44}$$

Condition (43) corresponds to lossless media where $\overleftrightarrow{\varepsilon}$ and $\overleftrightarrow{\mu}$ are Hermitian. In this case Ω^2 is known to be quasi-Hermitian [2]. Conditions (44) do not require Hermiticity of either of $\overleftrightarrow{\varepsilon}$ and $\overleftrightarrow{\mu}$, yet they imply quasi-Hermiticity of Ω^2. This is quite remarkable, because whenever (44) holds,

$$\lambda_\pm=\frac{\varepsilon_1}{\mu_1}\left[(\varepsilon_1\pm\alpha)^2+\gamma_\epsilon^2\right]^{-1}. \tag{45}$$

Therefore, $\lambda_\pm$ are real and positive. According to (37) – (40), this means that the amplitude of both the electric and magnetic fields are bounded biperiodic functions of time; *quasi-Hermiticity of* Ω^2 *confines the amplitude of the propagating electric and magnetic fields to a bounded region in the* x-y *plane that does not shrink in time.*

[5] According to (37), $\vec{E}(\vec{r},t)$ is given by a real vector-valued function of $\vec{x}$ and t, namely $\vec{n}_E(\vec{x},t)$ times a complex scalar function of $\vec{x}$, i.e., $\mathcal{E}e^{ik_3x}$. Hence we can associate a total phase factor to $\vec{E}(\vec{r},t)$, namely $\mathcal{E}e^{ik_3x}/|\mathcal{E}|$ which does not change in time. In view of (39), the same is true for $\vec{B}(\vec{r},t)$.

In view of (41) and (44), a necessary condition for the pseudo-Hermiticity of Ω^2 is the second equation appearing in both (41) and (44). For the usual material where μ_1 and ε_1 are positive, this equation implies that γ_ε and γ_μ have opposite sign. Therefore, to maintain the pseudo-Hermiticity of Ω^2 the material must simultaneously display loss with respect to the electric properties and gain with respect to the magnetic properties or vice versa.

As seen from (38) and (40), the propagating plane wave determined by the initial conditions (32) is not a time-harmonic solution of the Maxwell equations. It is rather a superposition of two time-harmonic plane-wave solutions. In general, the time-harmonic right-going, plane-wave solutions propagating along the z-axis has the following form.

$$\vec{E}_{\pm}^{(1)}(z,t) \quad = \quad A_{\pm}^{(1)} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} e^{i(k_3 z - \sqrt{\lambda_\pm}\,\omega_0 t)}, \qquad E_{\pm}^{(2)}(z,t) = A_{\pm}^{(2)} \begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} e^{i(k_3 z - \sqrt{\lambda_\pm}\,\omega_0 t)}, \tag{46}$$

where $A^{(1)}, A^{(2)} \in \mathbb{C}$. These solutions are clearly circularly polarized. For the cases that Ω^2 is quasi-Hermitian, i.e., either (43) or (44) holds, (46) are periodic solutions with a constant amplitude. For the cases that Ω^2 is pseudo-Hermitian but not quasi-Hermitian, either $\vec{E}_{+}^{(a)}$ or $\vec{E}_{-}^{(a)}$ is an exponentially decaying solution while the other is an exponentially growing solution.

Example 2: Consider the propagation of electromagnetic waves in a medium with complex symmetric permittivity and permeability tensors [11]:

$$\overset{\leftrightarrow}{\varepsilon} = \varepsilon_0 \overset{\leftrightarrow}{\Lambda}, \qquad \overset{\leftrightarrow}{\mu} = \mu_0 \overset{\leftrightarrow}{\Lambda}, \qquad \overset{\leftrightarrow}{\Lambda} := \begin{pmatrix} \mathfrak{a} & \mathfrak{g} & \mathfrak{u} \\ \mathfrak{g} & \mathfrak{b} & \mathfrak{h} \\ \mathfrak{u} & \mathfrak{h} & \mathfrak{c} \end{pmatrix}, \tag{47}$$

where $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{g}, \mathfrak{h}, \mathfrak{u} \in \mathbb{C}$. In this case, $\hat{\Omega}^2$ is a diagonalizable operator with a single nonzero eigenvalue, namely

$$\lambda_\pm = \lambda_0 := \frac{\mathfrak{a}\, k_1^2 + \mathfrak{n}\, k_2^3 + \mathfrak{c}\, k_3^2 + 2(\mathfrak{g}\, k_1 k_2 + \mathfrak{h}\, k_2 k_3 + \mathfrak{u}\, k_1 k_3)}{|\vec{k}|^2[\mathfrak{abc} + 2\mathfrak{ghu} - (\mathfrak{a}\mathfrak{h}^2 + \mathfrak{b}\mathfrak{u}^2 + \mathfrak{c}\mathfrak{g}^2)]}. \tag{48}$$

A direct consequence of this is the fact that pseudo-Hermiticity of Ω^2 coincides with its quasi-Hermiticity. The latter is achieved by requiring that λ_0 be real.

We can easily diagonalize $\hat{\Omega}^2$ by setting

$$S^{-1} = \begin{pmatrix} \frac{k_1}{k_3} & -\frac{\mathfrak{u}\, k_1 + \mathfrak{h}\, k_2 + \mathfrak{c}\, k_3}{\mathfrak{a}\, k_1 + \mathfrak{g}\, k_2 + \mathfrak{u}\, k_3} & -\frac{\mathfrak{g} k_1 + \mathfrak{b} k_2 + \mathfrak{h}\, k_3}{\mathfrak{a} k_1 + \mathfrak{g} k_2 + \mathfrak{u} k_3} \\ \frac{k_2}{k_3} & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \tag{49}$$

In the following we derive the expression for the time-harmonic plane-wave solutions of Maxwell's equations for this case. In view of (22) and (49), these are linear combinations of

$$\vec{E}^{(1)}(\vec{r},t) \quad = \quad \frac{1}{|\vec{k}|} \begin{pmatrix} -(\mathfrak{u} k_1 + \mathfrak{h} k_2 + \mathfrak{c} k_3) \\ 0 \\ \mathfrak{a} k_1 + \mathfrak{g} k_2 + \mathfrak{u} k_3 \end{pmatrix} (A^{(1+)} e^{i(\vec{k}\cdot\vec{r} - \sqrt{\lambda_0}\,\omega_0 t)} + A^{(1-)} e^{i(\vec{k}\cdot\vec{r} + \sqrt{\lambda_0}\,\omega_0 t)}), \tag{50}$$

$$\vec{E}^{(2)}(\vec{r},t) \quad = \quad \frac{1}{|\vec{k}|} \begin{pmatrix} -(\mathfrak{g} k_1 + \mathfrak{b} k_2 + \mathfrak{h} k_3) \\ \mathfrak{a} k_1 + \mathfrak{g} k_2 + \mathfrak{u} k_3 \\ 0 \end{pmatrix} (A^{(2+)} e^{i(\vec{k}\cdot\vec{r} - \sqrt{\lambda_0}\,\omega_0 t)} + A^{(2-)} e^{i(\vec{k}\cdot\vec{r} + \sqrt{\lambda_0}\,\omega_0 t)}), \tag{51}$$

where $A^{(1\pm)}$ and $A^{(2\pm)}$ are possibly $\vec{k}$-dependent complex coefficients.

Again whenever Ω^2 is quasi-Hermitian one obtains periodic time-harmonic solution. Otherwise, depending on the imaginary part of $\sqrt{\lambda_0}$ one may have an exponentially decaying or growing solution. The latter is clearly dependent on the magnitude and direction of $\vec{k}$. For example, for the special case of a plane wave propagating along the positive z-axis, we have

$$\vec{E}^{(1)}(z,t) = A^{(1+)} \begin{pmatrix} -\mathfrak{c} \\ 0 \\ \mathfrak{u} \end{pmatrix} e^{i(k_3 z - \sqrt{\lambda_0}\,\omega_0 t)}, \qquad \vec{E}^{(2)}(z,t) = A^{(2+)} \begin{pmatrix} -\mathfrak{h} \\ \mathfrak{u} \\ 0 \end{pmatrix} e^{i(k_3 z - \sqrt{\lambda_0}\,\omega_0 t)}, \quad (52)$$

where $\omega_0 = c\,k_3$ and

$$\lambda_0 = \left[\mathfrak{a}\mathfrak{b} + \frac{2\mathfrak{g}\mathfrak{h}\mathfrak{u}}{\mathfrak{c}} - \left(\frac{\mathfrak{a}\mathfrak{h}^2 + \mathfrak{b}\mathfrak{u}^2}{\mathfrak{c}} + \mathfrak{g}^2\right)\right]^{-1}. \quad (53)$$

For the special case, $\mathfrak{a} = \mathfrak{b} = (1+\mathfrak{u}^2)/\mathfrak{c}$, $\mathfrak{g} = \mathfrak{u}^2/\mathfrak{c}$, $\mathfrak{h} = \mathfrak{u}$, that is considered in [11], we have $\lambda_0 = \mathfrak{c}^2$. Therefore, Ω^2 is quasi-Hermitian and the plane wave solutions (52) do not decay in time provided that $\mathfrak{c}$ is real. They are exponentially decaying (growing) solutions, for $\mathrm{Im}(\mathfrak{c}) > 0$ ($\mathrm{Im}(\mathfrak{c}) < 0$).

Example 3: Consider a time-harmonic plane wave propagating along the positive z-axis in a medium with complex symmetric permittivity and permeability tensors:

$$\overleftrightarrow{\varepsilon} = \varepsilon_0 \begin{pmatrix} \mathfrak{f} - i\mathfrak{g} & \mathfrak{g} & 0 \\ \mathfrak{g} & \mathfrak{f} + i\mathfrak{g} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \overleftrightarrow{\mu} = \mu_0 \overleftrightarrow{1}, \quad (54)$$

where $\mathfrak{f}$ and $\mathfrak{g}$ are nonzero (possibly) complex parameters. In this case, $k_1 = k_2 = 0$, $\omega_0 = ck_3$, and we can easily show that $\hat{\Omega}^2$ is a non-diagonalizable operator, i.e., it is an example of Case 2 of Section 2. Moreover, we have

$$\lambda = \mathfrak{f}, \qquad S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \mathfrak{g} & i\,\mathfrak{g} & 0 \end{pmatrix}, \qquad S^{-1} = \begin{pmatrix} 0 & -i & \mathfrak{g}^{-1} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (55)$$

According to (26) the propagating electric field is given by

$$\vec{E}(z,t) = A^{+} \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} e^{i(k_3 z - \sqrt{\mathfrak{f}}\,\omega_0 t)}, \quad (56)$$

where $A^{+} \in \mathbb{C}$. Again Ω^2 is pseudo-Hermitian provided that $\mathfrak{f}$ is real. In this case, (56) is periodic in time. Otherwise, its amplitude is an exponentially decreasing or increasing function of time. These correspond to $\mathrm{Im}(\sqrt{\mathfrak{f}}) > 0$ and $\mathrm{Im}(\sqrt{\mathfrak{f}}) < 0$, respectively.

Another peculiarity of the model considered here is that the expression (56) for the time-harmonic plane-wave solution does not involve $\mathfrak{g}$. We only require that $\mathfrak{g}$ takes a nonzero value. Furthermore, (56) describes a right-going clockwise circularly polarized field. The non-diagonalizability of Ω^2 implies that there is no other time-harmonic plane wave solutions

propagating along the positive z-axis that is linearly independent of (56). We may take this as an indication of the "left-handedness" of the medium. This observation motivates the solution of the Maxwell's equation associated with the following initial conditions.

$$\vec{E}_0(\vec{r}) = \mathcal{E}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} e^{ik_3 z}, \qquad \vec{B}_0(\vec{r}) = \vec{0}. \tag{57}$$

The result reads

$$\vec{E}(\vec{r},t) = \vec{E}(z,t) = \mathcal{E}\begin{pmatrix} \cos(\sqrt{\mathfrak{f}}\,\omega_0 t) + \frac{i\mathfrak{g}\,\omega_0 t}{\sqrt{\mathfrak{f}}}\sin(\sqrt{\mathfrak{f}}\,\omega_0 t) \\ -i\cos(\sqrt{\mathfrak{f}}\,\omega_0 t) - \frac{\mathfrak{g}\,\omega_0 t}{\sqrt{\mathfrak{f}}}\sin(\sqrt{\mathfrak{f}}\,\omega_0 t) \\ 0 \end{pmatrix} e^{ik_3 z}, \tag{58}$$

$$\vec{B}(\vec{r},t) = \vec{B}(z,t) = c^{-1}\mathcal{E}\begin{pmatrix} 1 - \frac{i\mathfrak{g}\,\omega_0^2 t^2}{2} \\ -i(1 + \frac{i\mathfrak{g}\,\omega_0^2 t^2}{2}) \\ 0 \end{pmatrix} \frac{\sin(\sqrt{\mathfrak{f}}\,\omega_0 t) e^{ik_3 z}}{\sqrt{\mathfrak{f}}}, \tag{59}$$

where we have made use of (13), (17), (24), (25), and (4). The appearance of the factors $\omega_0 t$ and $\omega_0^2 t^2$ as coefficients of the period functions of t in (58) and (59) is a clear indication that these solutions are not time-harmonic.[6] Indeed, even for the cases that $\mathfrak{f}$ is real, these solutions are unbounded functions of time. This in turn means that one cannot maintain right-handed circularly polarized initial waves of the form (57) in such a material.[7]

4 Concluding Remarks

The information on the propagation of electromagnetic waves in a stationary, possibly inhomogeneous, and anisotropic media is encoded in a generally matrix-valued differential operator Ω^2. In the absence of loss and gain, this operator is quasi-Hermitian and one can use the properties of quasi-Hermitian operator to obtain the solution of the wave equation. In the presence of gain or loss, there is no guarantee that this operator is diagonalizable or has real eigenvalues.

In this article we examined the Jordan decomposition of this operator for a class of anisotropic active media and showed how the spectral properties of Ω^2 might be used in the description of the propagating waves. In particular, we examined the time-harmonic and plane-wave solutions for three class of toy models, derived explicit conditions on the permittivity and permeability tensors that would render Ω^2 pseudo-Hermitian or quasi-Hermitian, and offered a physical interpretation of the pseudo-Hermiticity and quasi-Hermiticity of Ω^2 in terms of the behavior of the propagating plane-waves solutions.

An interesting observation we made was that Ω^2 might actually be non-diagonalizable. We constructed an explicit model with this property and demonstrated a surprising feature of this model that could be interpreted as its left-handedness. Whether such exotic material can exist in nature or be manufactured is a subject of a separate investigation.

Our results may be extended in at least the following two main directions. Firstly, one may try to generalize the method to non-homogeneous isotropic media. This can be achieved using

[6] Note that $\mathfrak{g} \neq 0$.

[7] There is not reason to believe that such exotic material can exist in real life.

the methods of Fourier analysis. Secondly, one might attempt to include the effects of dispersion following the ideas presented in [6].

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